SCIENCE
PASSION
TECHNOLOGY

## Logic and Computability

# Natural Deduction for Predicate Logic

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https://xkcd.com/2497/

# Time Line - Topics

## Propositional Logic

- Syntax & Semantic
- Natural Deduction
- Decide Satisfiability (DPLL)
- Equivalence Checking / Normal forms
- Data structures (BDDs)
- Introduction to Z3

## Predicate Logic

- Syntax & Semantic
- Natural deduction
- Satisfiability Modulo Theory (SMT)
- Decide Satisfiability (DPLL(T))
- Solving Problems via Z3

# Motivation

- Extend Natural Deduction to Predicate Logic
  - Richer Language → More powerful proofs
- Basis for “Real Proofs”

# Plan for Today

- New Rules for Natural Deduction
  - ∀-Quantifier
    - Rules for introduction and elimination
  - ∃-Quantifier
    - Rules for introduction and elimination
- Construct natural deduction proofs
  - Many examples
- Counterexample to proof that sequents are invalid

# Learning Outcomes

After this lecture…

1. for valid sequents in predicate logic, students can construct natural deduction proofs to prove that the sequent is valid.
2. for invalid sequents in predicate logic, students can construct counter examples to show that the sequent is invalid.

# Proof Rules for Universal Quantification

$$\boxed{\frac{\forall x\ \varphi}{\varphi\,[t/x]}\ \forall_e}$$

$\forall x\ \varphi$ is true, we are allowed to replace the $x$ in $\varphi$ with any term $t$.

**Substitution $\varphi[t/x]$**

Term Variable

- Reads: „$\varphi$ with $t$ for $x$„
- Example:

$$\frac{\forall x\ \big(P(f(x,y)) \lor Q(x)\big)}{P(f(a,y)) \lor Q(a)}\ \forall e \qquad \text{with } \varphi[a/x]$$

# Proof Rules for Universal Quantification

$$\boxed{\dfrac{\forall x\ \varphi}{\varphi\,[t/x]}\ \forall_e}$$

$\forall x\ \varphi$ is true, we are allowed to replace the $x$ in $\varphi$ with any term $t$.

**Substitution $\varphi[t/x]$**

Conditions for Substitution

1. Replace only *free* variables
   - $\varphi = \exists y\big(P(x, y) \vee Q(y)\big)$

     bound
   - $\varphi[a/y] = \varphi$

# Proof Rules for Universal Quantification

$$\frac{\forall x \ \varphi}{\varphi\,[t/x]}\ \forall_e$$

$\forall x\ \varphi$ is true, we are allowed to replace the $x$ in $\varphi$ with any term $t$.

**Substitution $\varphi[t/x]$**

Conditions for Substitution

1. Replace only *free* variables
2. The term $t$ must be *free* for a variable x → No capturing
   - $\varphi = \exists x(P(x) \lor Q(z))$
     free
   - $\varphi[f(x)/z] = \exists x\left(P(x) \lor Q(f(x))\right)$
     bound

# Proof Rules for Universal Quantification

$$\boxed{\dfrac{\forall x\ \varphi}{\varphi\,[t/x]}\ \forall_e}$$

$\forall x\ \varphi$ is true, we are allowed to replace the $x$ in $\varphi$ with any term $t$.

**Substitution $\varphi[t/x]$**

Conditions for Substitution

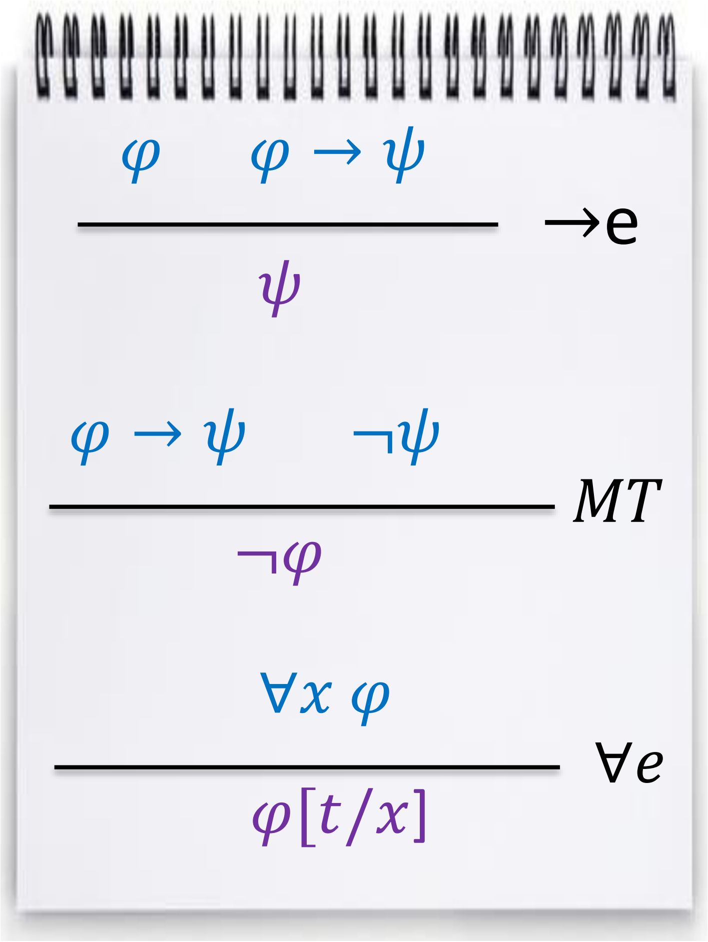
1. Replace only *free* variables
2. The term $t$ must be *free* for a variable x → No capturing
   - $\varphi = \exists x(P(x) \vee Q(z))$

     ↳ free
   - $\varphi[f(x)/z] = \exists x(P(x) \vee Q(z))$

     ↳ free

# Example 1

- $\forall x\big(\neg P(x) \to Q(x)\big),\ \neg Q(t) \vdash P(t)$



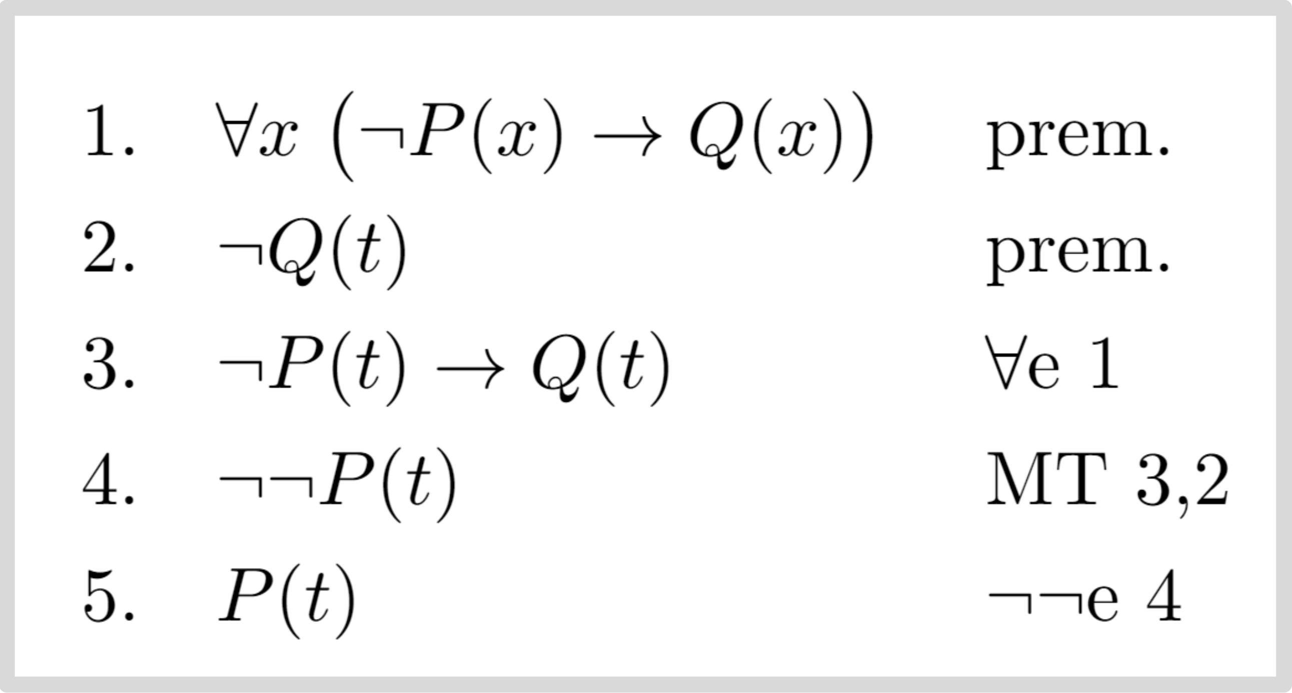
$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ \to\mathrm{e}$$

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}\ MT$$

$$\frac{\forall x\ \varphi}{\varphi[t/x]}\ \forall e$$

# Example 1

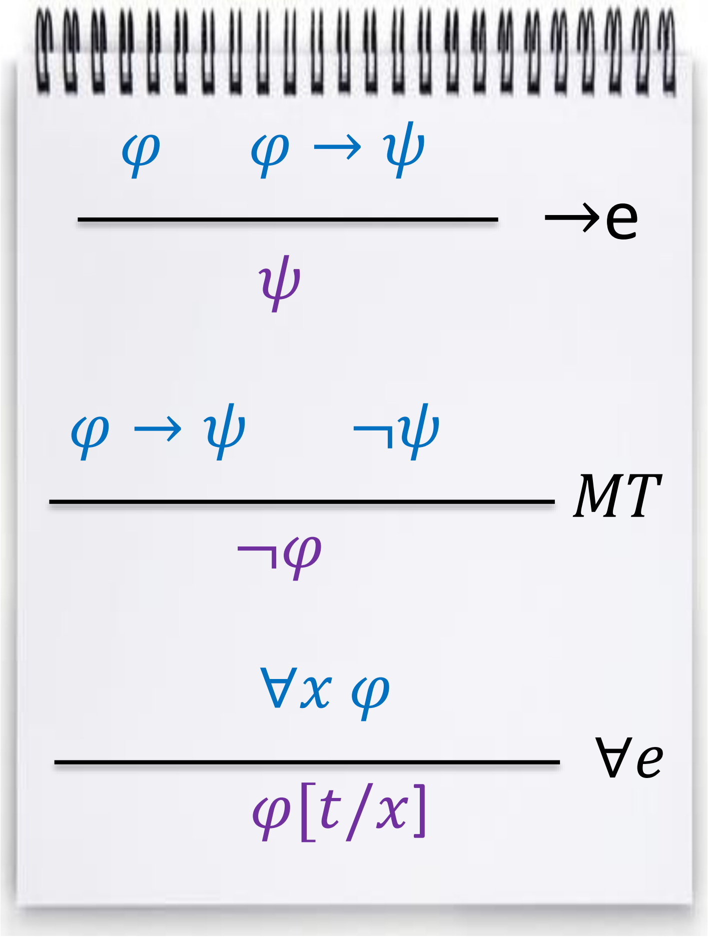
- $\forall x(\neg P(x) \to Q(x)),\ \neg Q(t) \vdash P(t)$



| | | |
|---|---|---|
| 1. | $\forall x\,(\neg P(x) \to Q(x))$ | prem. |
| 2. | $\neg Q(t)$ | prem. |
| 3. | $\neg P(t) \to Q(t)$ | $\forall$e 1 |
| 4. | $\neg\neg P(t)$ | MT 3,2 |
| 5. | $P(t)$ | $\neg\neg$e 4 |

# Example 2

- $\forall x P(x) \land \forall x (P(y) \to Q(x)) \vdash \quad Q(z)$



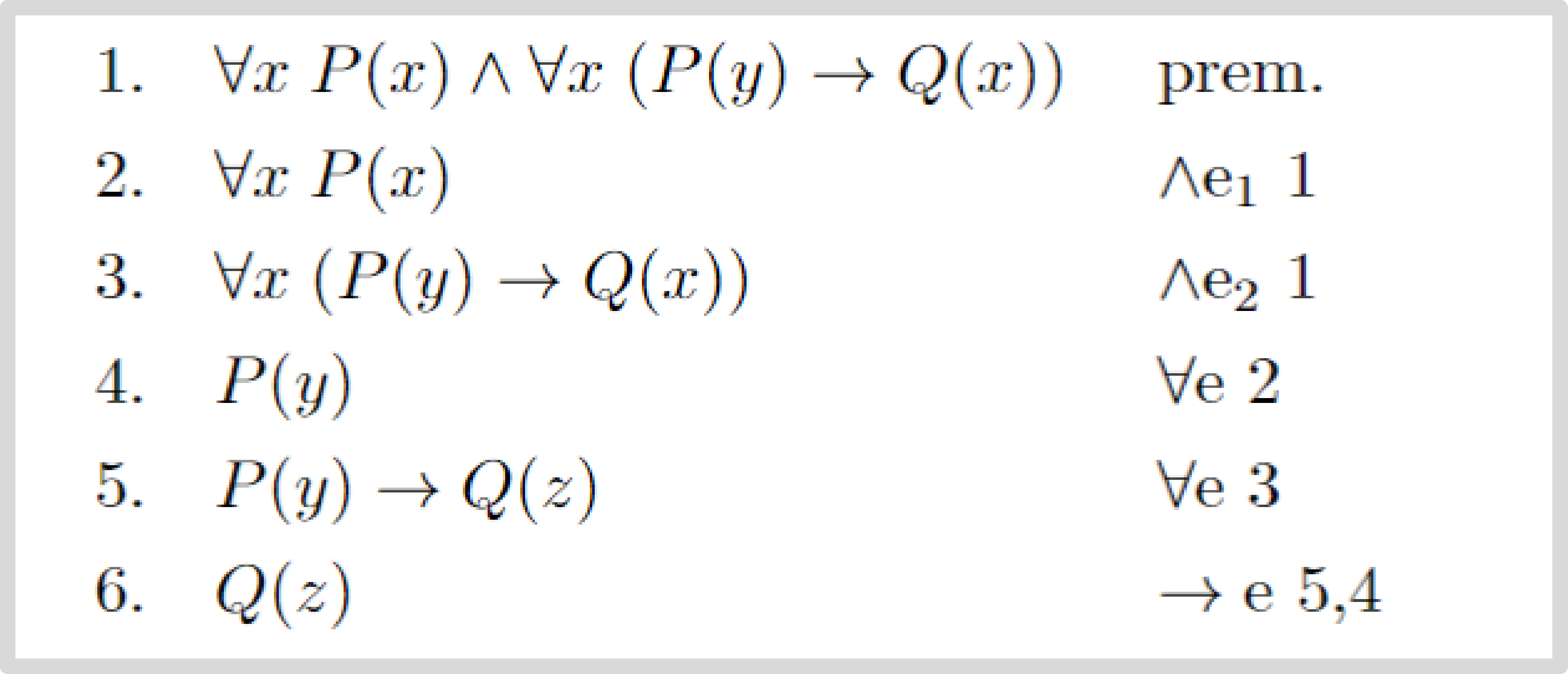
$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to\mathrm{e}$$

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi} MT$$

$$\frac{\forall x\ \varphi}{\varphi[t/x]} \forall e$$

# Example 2

- $\forall x P(x) \land \forall x (P(y) \rightarrow Q(x)) \vdash \quad Q(z)$



| | | |
|---|---|---|
| 1. | $\forall x\ P(x) \land \forall x\ (P(y) \rightarrow Q(x))$ | prem. |
| 2. | $\forall x\ P(x)$ | $\land e_1$ 1 |
| 3. | $\forall x\ (P(y) \rightarrow Q(x))$ | $\land e_2$ 1 |
| 4. | $P(y)$ | $\forall e$ 2 |
| 5. | $P(y) \rightarrow Q(z)$ | $\forall e$ 3 |
| 6. | $Q(z)$ | $\rightarrow e$ 5,4 |

# Proof Rules for Universal Quantification

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad\varphi\,[x_0/x] \end{array}} \quad x_0 \text{ fresh}}{\forall x\ \varphi}\ \forall_i$$

- If we can proof $\varphi[x_0/x]$ for a **fresh variable $x_0$**, we can derive $\forall x\ \varphi$!

# Example 3

- $\forall x\big(P(x) \rightarrow Q(x)\big),\ \forall x P(x) \vdash \quad \forall x Q(x)$

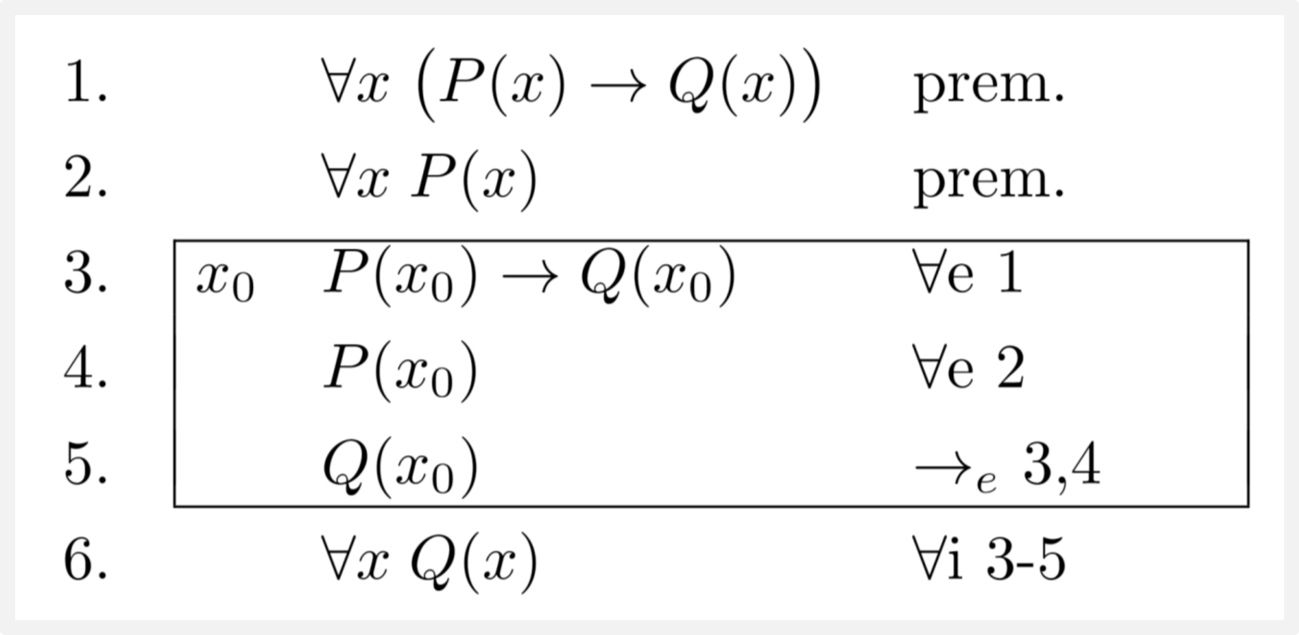
$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad\varphi\left[x_0/x\right] \end{array}} \quad x_0 \text{ fresh}}{\forall x\ \varphi}\ \forall_i$$

$$\dfrac{\forall x\ \varphi}{\varphi\left[t/x\right]}\ \forall_e$$

$$\dfrac{\varphi \qquad \varphi \rightarrow \psi}{\psi}\ \rightarrow_e$$

# Example 3

- $\forall x \big(P(x) \rightarrow \ Q(x)\big), \ \ \forall x P(x) \ \vdash \quad \forall x Q(x)$



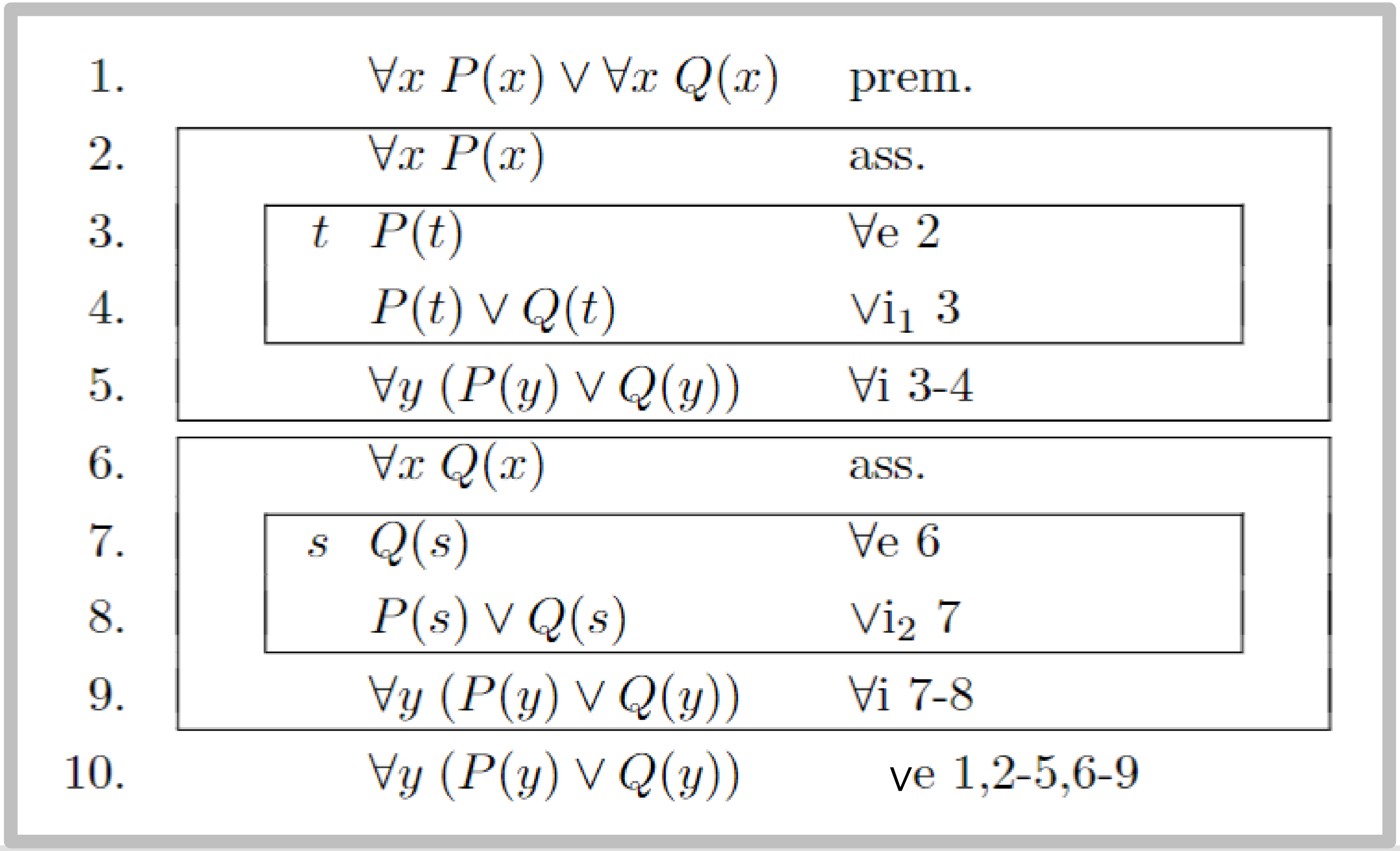
| | | | |
|---|---|---|---|
| 1. | | $\forall x \ (P(x) \rightarrow Q(x))$ | prem. |
| 2. | | $\forall x \ P(x)$ | prem. |
| 3. | $x_0$ | $P(x_0) \rightarrow Q(x_0)$ | $\forall$e 1 |
| 4. | | $P(x_0)$ | $\forall$e 2 |
| 5. | | $Q(x_0)$ | $\rightarrow_e$ 3,4 |
| 6. | | $\forall x \ Q(x)$ | $\forall$i 3-5 |

# Example 4

- $\forall x P(x) \lor \forall x\, Q(x) \quad \vdash \quad \forall y(\, P(y) \lor Q(y))$

# Example 4

- $\forall x P(x) \vee \forall x\, Q(x) \;\vdash\; \forall y(\, P(y) \vee Q(y))$



| | | | |
|---|---|---|---|
| 1. | | $\forall x\; P(x) \vee \forall x\; Q(x)$ | prem. |
| 2. | | $\forall x\; P(x)$ | ass. |
| 3. | $t$ | $P(t)$ | $\forall$e 2 |
| 4. | | $P(t) \vee Q(t)$ | $\vee$i$_1$ 3 |
| 5. | | $\forall y\; (P(y) \vee Q(y))$ | $\forall$i 3-4 |
| 6. | | $\forall x\; Q(x)$ | ass. |
| 7. | $s$ | $Q(s)$ | $\forall$e 6 |
| 8. | | $P(s) \vee Q(s)$ | $\vee$i$_2$ 7 |
| 9. | | $\forall y\; (P(y) \vee Q(y))$ | $\forall$i 7-8 |
| 10. | | $\forall y\; (P(y) \vee Q(y))$ | ve 1,2-5,6-9 |

# Proof Rules for Existential Quantification

$$\boxed{\dfrac{\varphi\,[t/x]}{\exists x\ \varphi}\ \exists_i}$$

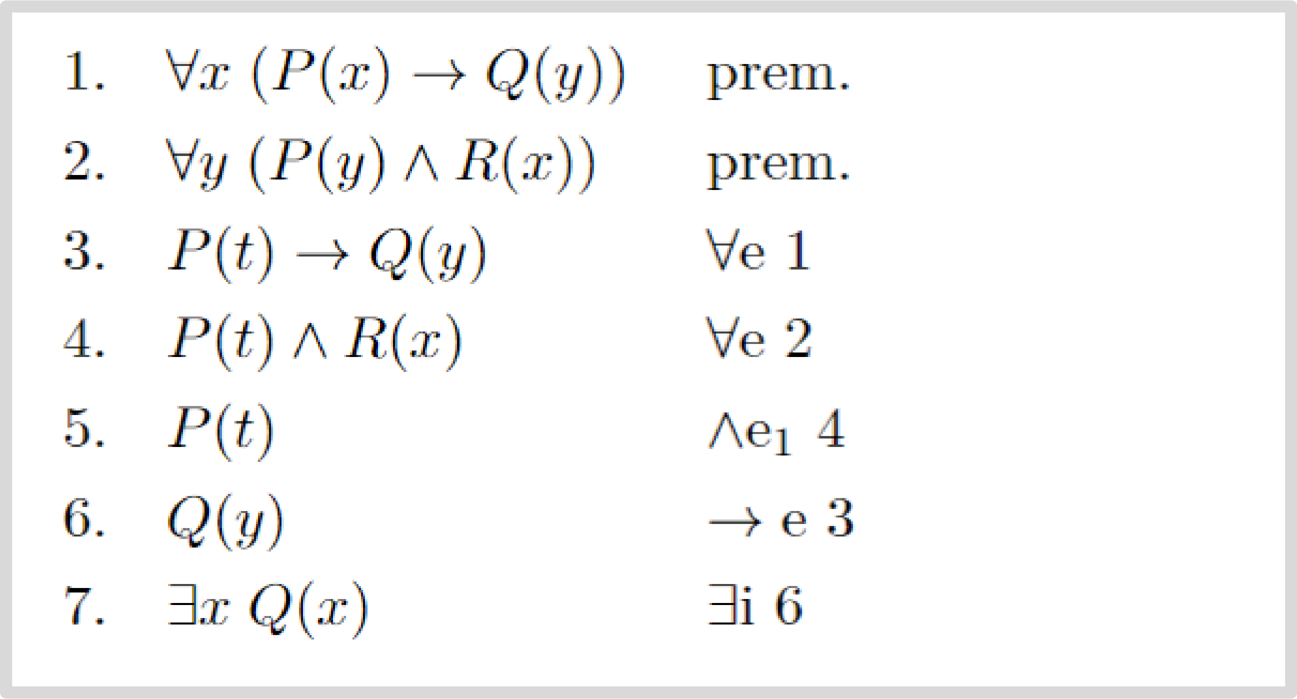
- $\exists x$ only asks for $\varphi$ to be true for some term $t$
- Side condition: that $t$ be *free* for x in $\varphi$

# Example 5

- $\forall x\big(P(x) \rightarrow\ Q(y)\big),\ \ \forall y\big(P(y) \wedge R(x)\big)\ \ \vdash\ \ \ \exists x\ Q(x)$

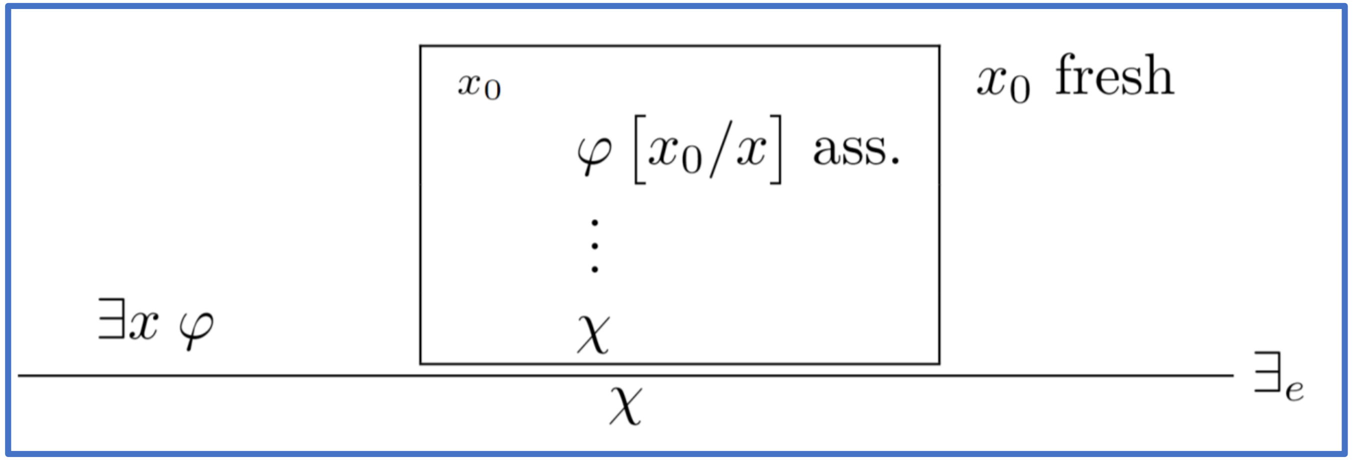
# Example 5

- $\forall x(P(x) \to Q(y)),\ \forall y(P(y) \wedge R(x)) \vdash \exists x\, Q(x)$



| | | |
|---|---|---|
| 1. | $\forall x\ (P(x) \to Q(y))$ | prem. |
| 2. | $\forall y\ (P(y) \wedge R(x))$ | prem. |
| 3. | $P(t) \to Q(y)$ | $\forall$e 1 |
| 4. | $P(t) \wedge R(x)$ | $\forall$e 2 |
| 5. | $P(t)$ | $\wedge \mathrm{e}_1$ 4 |
| 6. | $Q(y)$ | $\to$ e 3 |
| 7. | $\exists x\ Q(x)$ | $\exists$i 6 |

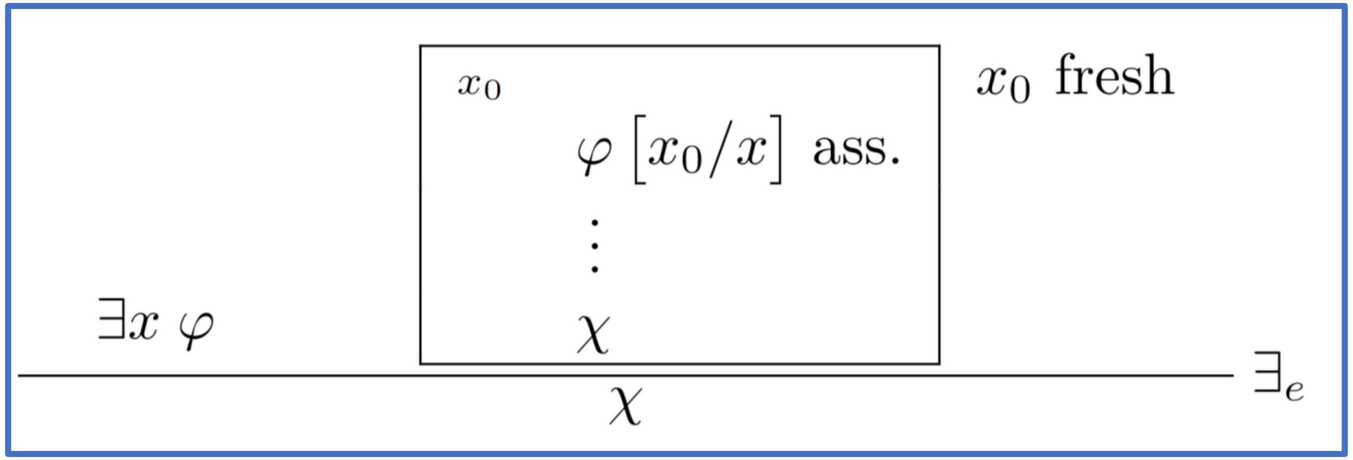
# Proof Rules for Existential Quantification



$$\frac{\exists x\ \varphi \qquad \boxed{\begin{array}{l} x_0 \\ \qquad \varphi\left[x_0/x\right] \ \text{ass.} \\ \qquad \vdots \\ \qquad \chi \end{array}} \quad x_0 \text{ fresh}}{\chi}\ \exists_e$$

- From $\exists x\ \varphi$, we know that $\varphi$ is true for at least one value of $x$
- If we can prove $\chi$ without the exact knowledge of the value $x$, then $\chi$ can be deduced simply from the fact that there exists a value for $x$.
  - → thus we use a fresh variable $x_0$ in the proof
  - If by assuming $\varphi[x_0/x]$, we can prove $\chi$ inside the box, then $\chi$ can be deduced outside of the box

# Proof Rules for Existential Quantification



- From $\exists x\ \varphi$, we know that $\varphi$ is true for at least one value of $x$
- If we can proof $\chi$ without the exact knowledge of the value $x_0$, then $\chi$ can be deduced simply from the fact that there exists an $x_0$.

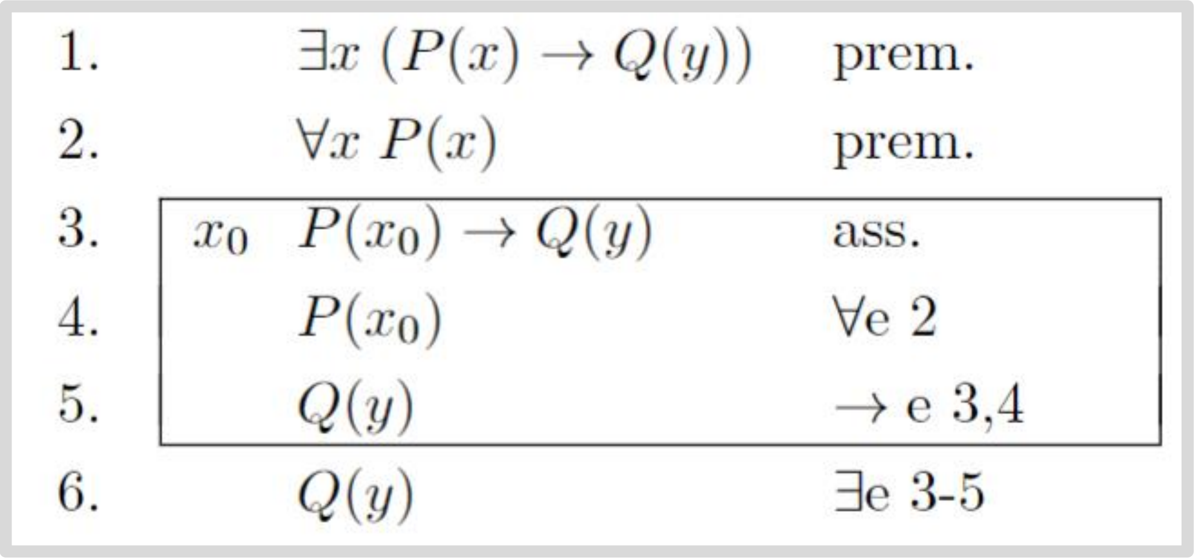
- **Important: $\chi$ is not allowed to contain $x_0$!**

# Example 6

- $\exists x \big( P(x) \rightarrow \; Q(y) \big), \;\; \forall x P(x) \;\; \vdash \quad Q(y)$

# Example 6

- $\exists x\big(P(x) \to Q(y)\big),\ \forall x P(x) \vdash Q(y)$



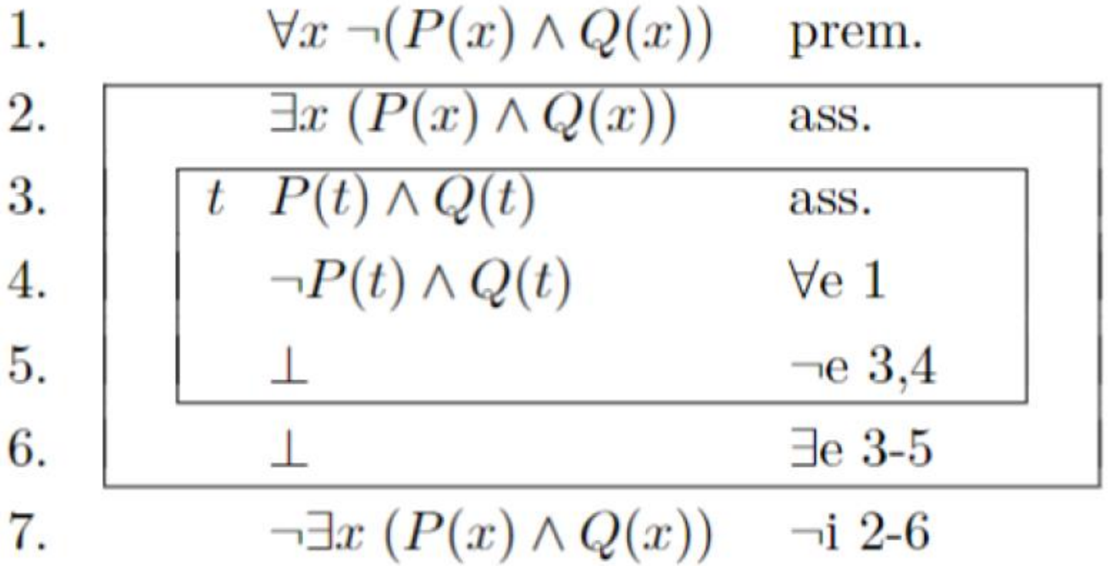
| | | | |
|---|---|---|---|
| 1. | | $\exists x\ (P(x) \to Q(y))$ | prem. |
| 2. | | $\forall x\ P(x)$ | prem. |
| 3. | $x_0$ | $P(x_0) \to Q(y)$ | ass. |
| 4. | | $P(x_0)$ | $\forall$e 2 |
| 5. | | $Q(y)$ | $\to$e 3,4 |
| 6. | | $Q(y)$ | $\exists$e 3-5 |

# Example 7

- $\forall x \neg \big(P(x) \land Q(x)\big) \vdash \neg \exists x\ (P(x) \land Q(x))$

# Example 7

- $\forall x \neg (P(x) \wedge Q(x)) \vdash \neg \exists x \, (P(x) \wedge Q(x))$



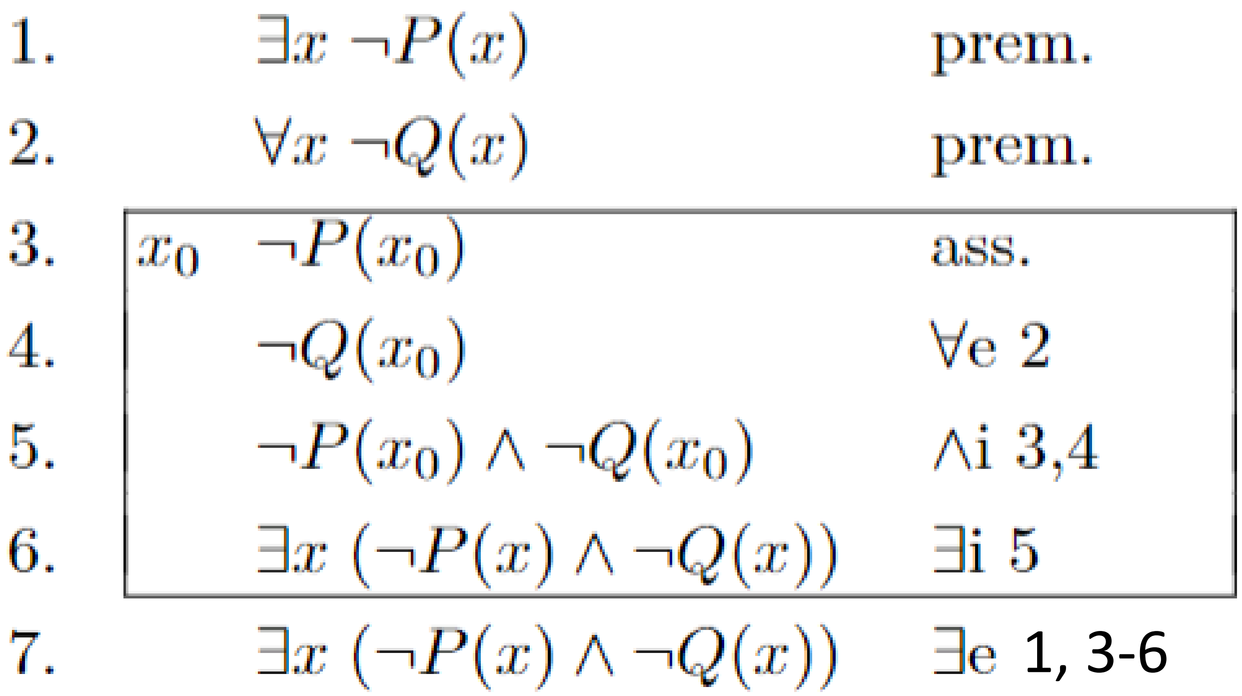
| | | | |
|---|---|---|---|
| 1. | | $\forall x \, \neg(P(x) \wedge Q(x))$ | prem. |
| 2. | | $\exists x \, (P(x) \wedge Q(x))$ | ass. |
| 3. | $t$ | $P(t) \wedge Q(t)$ | ass. |
| 4. | | $\neg P(t) \wedge Q(t)$ | $\forall$e 1 |
| 5. | | $\perp$ | $\neg$e 3,4 |
| 6. | | $\perp$ | $\exists$e 3-5 |
| 7. | | $\neg \exists x \, (P(x) \wedge Q(x))$ | $\neg$i 2-6 |

# Example 8

- $\exists x \neg P(x), \quad \forall x \neg Q(x) \vdash \quad \exists x\,(\neg P(x) \wedge \neg Q(x))$

# Example 8

- $\exists x \neg P(x), \quad \forall x \neg Q(x) \vdash \quad \exists x\,(\neg P(x) \wedge \neg Q(x))$



| | | | |
|---|---|---|---|
| 1. | | $\exists x\ \neg P(x)$ | prem. |
| 2. | | $\forall x\ \neg Q(x)$ | prem. |
| 3. | $x_0$ | $\neg P(x_0)$ | ass. |
| 4. | | $\neg Q(x_0)$ | $\forall$e 2 |
| 5. | | $\neg P(x_0) \wedge \neg Q(x_0)$ | $\wedge$i 3,4 |
| 6. | | $\exists x\ (\neg P(x) \wedge \neg Q(x))$ | $\exists$i 5 |
| 7. | | $\exists x\ (\neg P(x) \wedge \neg Q(x))$ | $\exists$e 1, 3-6 |

# Invalid Sequents

$$\exists x\big(P(x) \rightarrow Q(x)\big), \qquad \exists x P(x) \quad \vdash \quad Q(y)$$

# Invalid Sequents

$$\exists x\big(P(x) \to Q(y)\big), \qquad \exists x P(x) \quad \vdash \quad Q(y)$$

✓ ✓ ✗

- Model $M$:
  - $A \;=\; \{a, b\}$
  - $P^M = \{a\}$
  - $Q^M = \{a\}$
  - $y \leftarrow b$

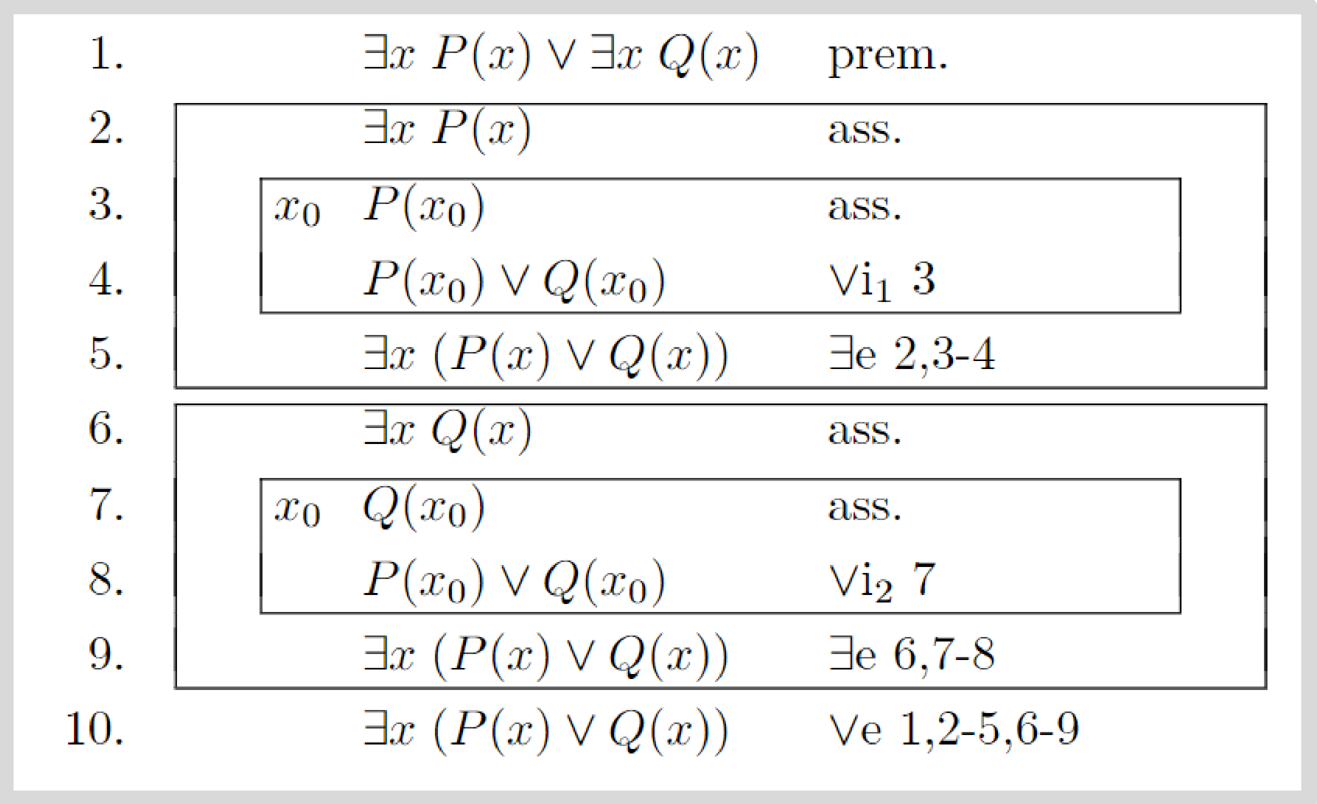
$$\left.\begin{array}{l} M \vDash \exists x\big(P(x) \to Q(y)\big),\ \exists x\, P(x) \\ M \nvDash Q(y) \end{array}\right\} \; M \text{ is a counterexample}$$

# Example 9

6.1.33 Consider the following natural deduction proof for the sequent

$$\exists x\ P(x) \lor \exists x\ Q(x) \quad \vdash \quad \exists x\ (P(x) \lor Q(x)).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.



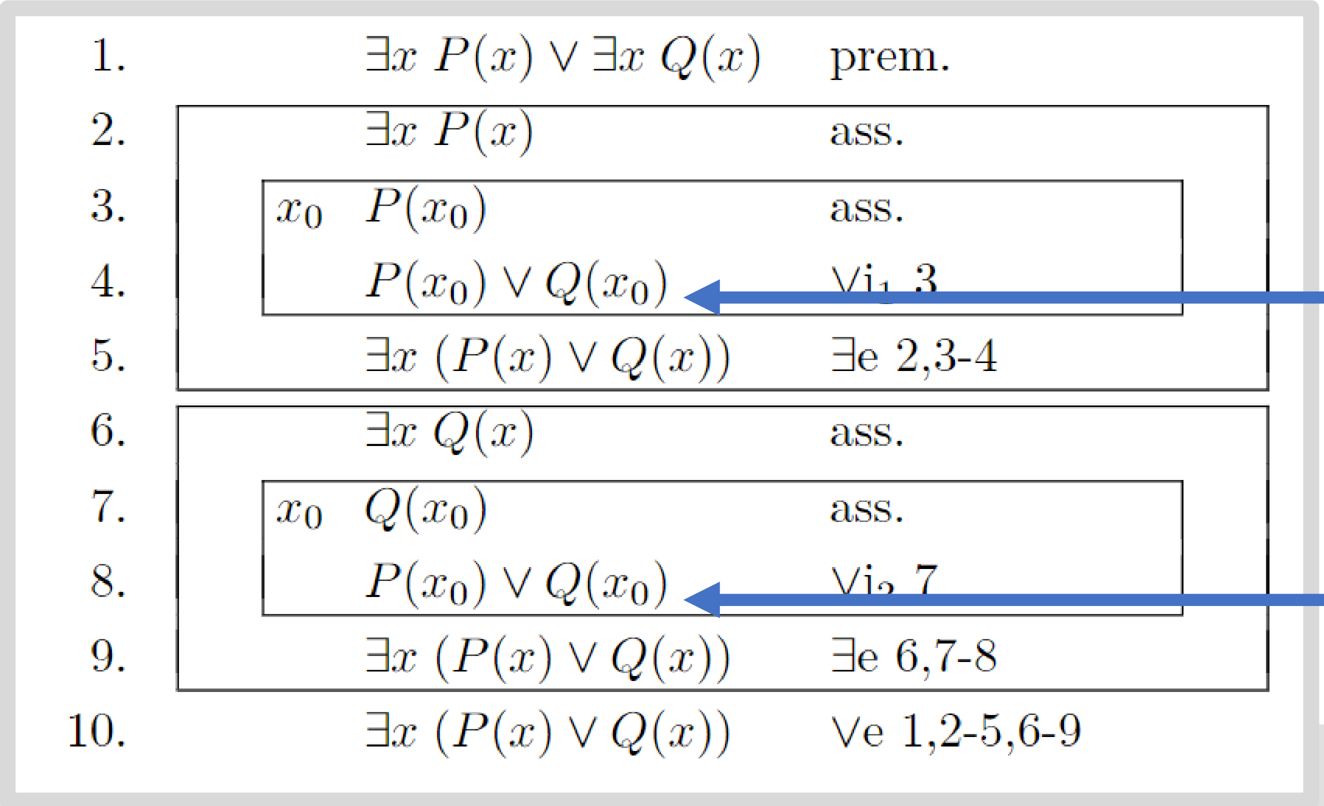
| | | | |
|---|---|---|---|
| 1. | | $\exists x\ P(x) \lor \exists x\ Q(x)$ | prem. |
| 2. | | $\exists x\ P(x)$ | ass. |
| 3. | $x_0$ | $P(x_0)$ | ass. |
| 4. | | $P(x_0) \lor Q(x_0)$ | $\lor$i$_1$ 3 |
| 5. | | $\exists x\ (P(x) \lor Q(x))$ | $\exists$e 2,3-4 |
| 6. | | $\exists x\ Q(x)$ | ass. |
| 7. | $x_0$ | $Q(x_0)$ | ass. |
| 8. | | $P(x_0) \lor Q(x_0)$ | $\lor$i$_2$ 7 |
| 9. | | $\exists x\ (P(x) \lor Q(x))$ | $\exists$e 6,7-8 |
| 10. | | $\exists x\ (P(x) \lor Q(x))$ | $\lor$e 1,2-5,6-9 |

# Example 9

6.1.33 Consider the following natural deduction proof for the sequent

$$\exists x\, P(x) \vee \exists x\, Q(x) \quad \vdash \quad \exists x\, (P(x) \vee Q(x)).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

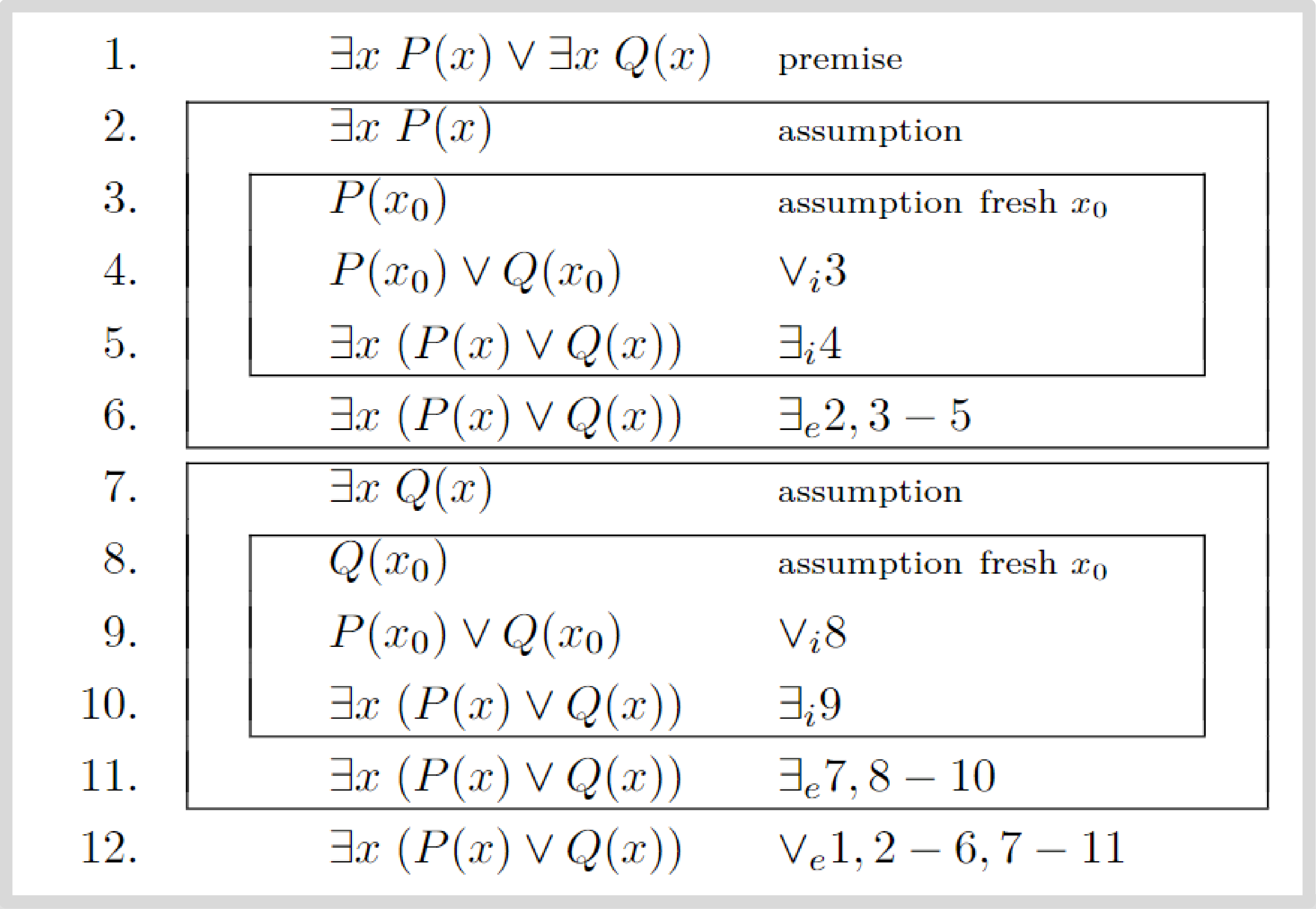


| | | | |
|---|---|---|---|
| 1. | | $\exists x\, P(x) \vee \exists x\, Q(x)$ | prem. |
| 2. | | $\exists x\, P(x)$ | ass. |
| 3. | $x_0$ | $P(x_0)$ | ass. |
| 4. | | $P(x_0) \vee Q(x_0)$ | $\vee i_1$ 3 |
| 5. | | $\exists x\, (P(x) \vee Q(x))$ | $\exists$e 2,3-4 |
| 6. | | $\exists x\, Q(x)$ | ass. |
| 7. | $x_0$ | $Q(x_0)$ | ass. |
| 8. | | $P(x_0) \vee Q(x_0)$ | $\vee i_2$ 7 |
| 9. | | $\exists x\, (P(x) \vee Q(x))$ | $\exists$e 6,7-8 |
| 10. | | $\exists x\, (P(x) \vee Q(x))$ | $\vee$e 1,2-5,6-9 |

$\exists i$ missing

$\exists i$ missing

# Example 9



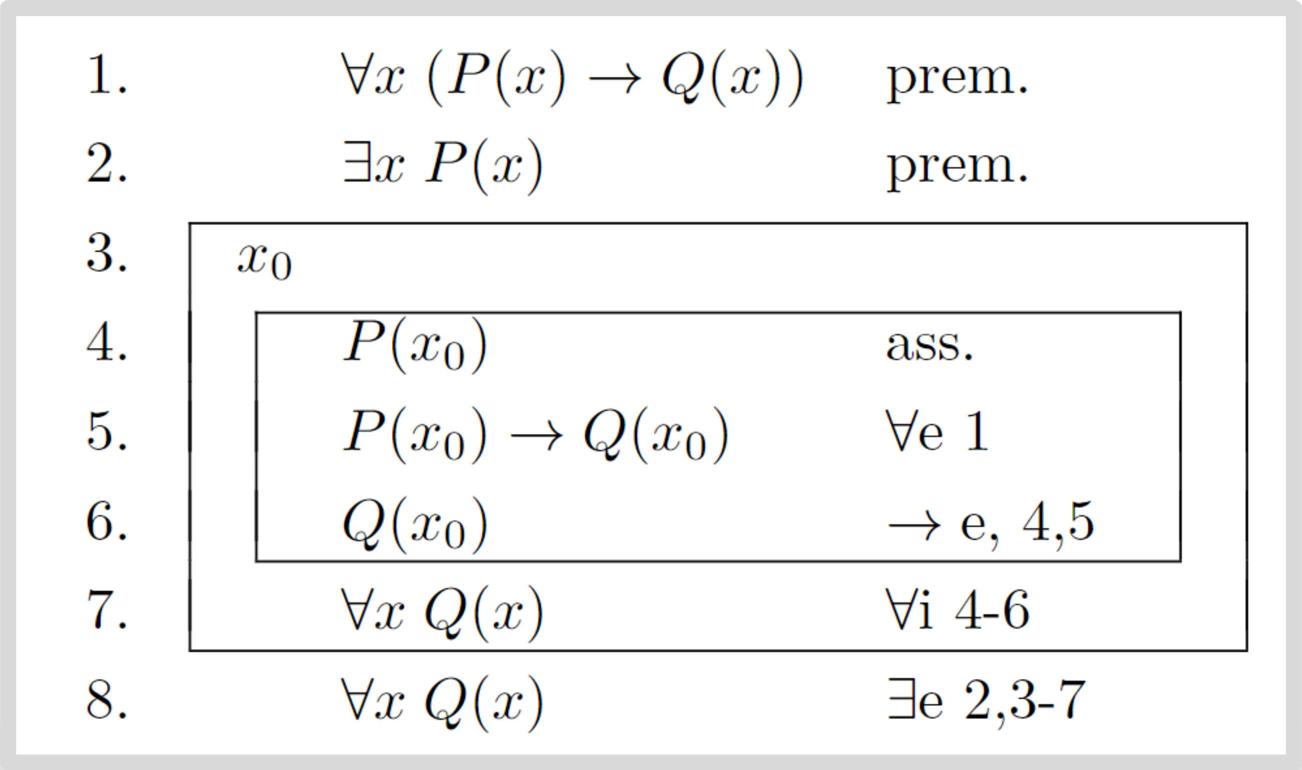
| | | |
|---|---|---|
| 1. | $\exists x\ P(x) \lor \exists x\ Q(x)$ | premise |
| 2. | $\exists x\ P(x)$ | assumption |
| 3. | $P(x_0)$ | assumption fresh $x_0$ |
| 4. | $P(x_0) \lor Q(x_0)$ | $\lor_i 3$ |
| 5. | $\exists x\ (P(x) \lor Q(x))$ | $\exists_i 4$ |
| 6. | $\exists x\ (P(x) \lor Q(x))$ | $\exists_e 2, 3-5$ |
| 7. | $\exists x\ Q(x)$ | assumption |
| 8. | $Q(x_0)$ | assumption fresh $x_0$ |
| 9. | $P(x_0) \lor Q(x_0)$ | $\lor_i 8$ |
| 10. | $\exists x\ (P(x) \lor Q(x))$ | $\exists_i 9$ |
| 11. | $\exists x\ (P(x) \lor Q(x))$ | $\exists_e 7, 8-10$ |
| 12. | $\exists x\ (P(x) \lor Q(x))$ | $\lor_e 1, 2-6, 7-11$ |

# Example 10

6.1.7 Consider the following natural deduction proof for the sequent

$$\forall x\,(P(x) \to Q(x)),\quad \exists x\,P(x) \quad\vdash\quad \forall x Q(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.



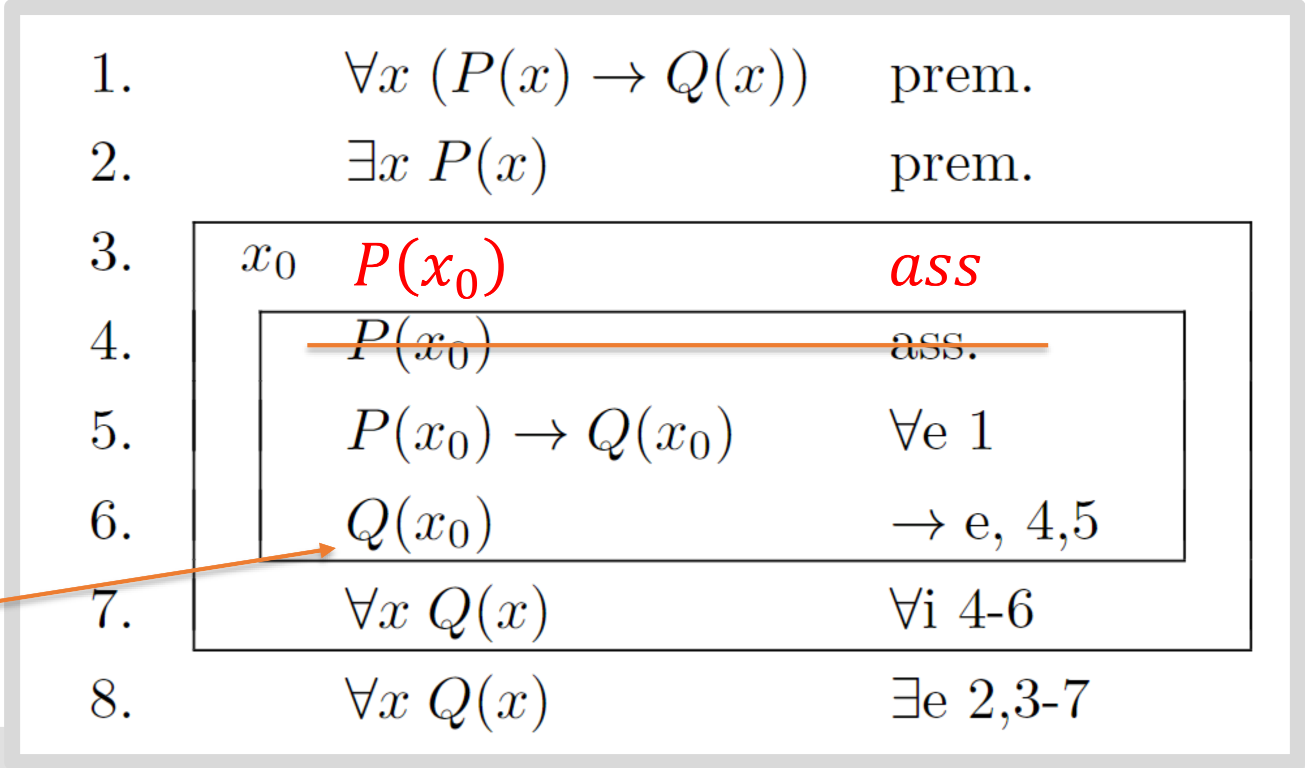
| | | |
|---|---|---|
| 1. | $\forall x\,(P(x) \to Q(x))$ | prem. |
| 2. | $\exists x\,P(x)$ | prem. |
| 3. | $x_0$ | |
| 4. | $P(x_0)$ | ass. |
| 5. | $P(x_0) \to Q(x_0)$ | ∀e 1 |
| 6. | $Q(x_0)$ | → e, 4,5 |
| 7. | $\forall x\,Q(x)$ | ∀i 4-6 |
| 8. | $\forall x\,Q(x)$ | ∃e 2,3-7 |

# Example 10

6.1.7 Consider the following natural deduction proof for the sequent

$$\forall x\,(P(x) \to Q(x)),\quad \exists x\,P(x) \quad \vdash \quad \forall x Q(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.



| | | | |
|---|---|---|---|
| 1. | | $\forall x\,(P(x) \to Q(x))$ | prem. |
| 2. | | $\exists x\,P(x)$ | prem. |
| 3. | $x_0$ | $P(x_0)$ | *ass* |
| 4. | | ~~$P(x_0)$~~ | ~~ass.~~ |
| 5. | | $P(x_0) \to Q(x_0)$ | ∀e 1 |
| 6. | | $Q(x_0)$ | → e, 4,5 |
| 7. | | $\forall x\,Q(x)$ | ∀i 4-6 |
| 8. | | $\forall x\,Q(x)$ | ∃e 2,3-7 |

**$x_0$ is not fresh for ∀i**

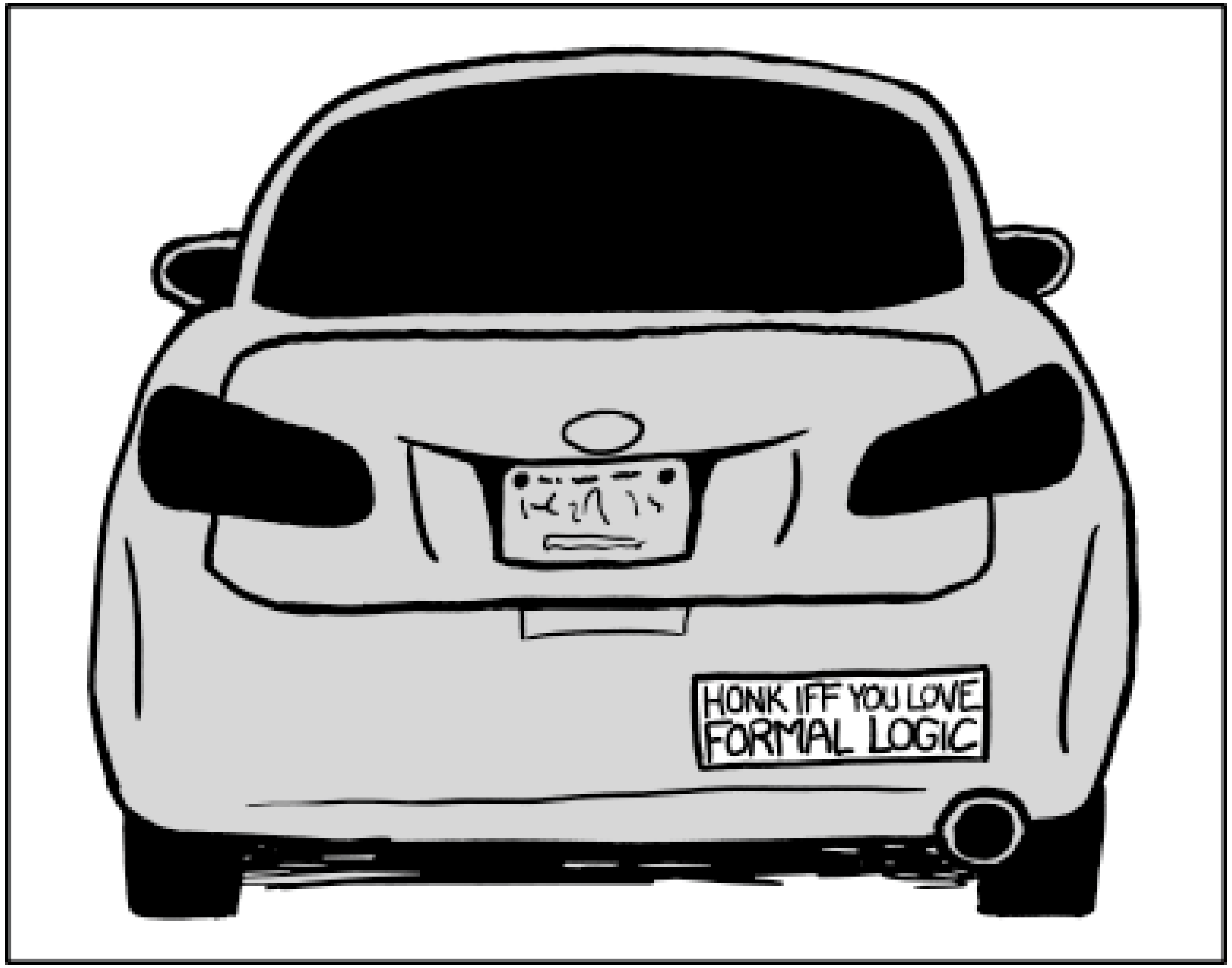
# Example 10

$$\forall x\big(P(x) \rightarrow Q(x)\big), \qquad \exists x P(x) \quad \vdash \quad \forall x Q(x)$$

- Model $M$:
    - $A \;=\; \{a, b\}$
    - $P^M = \{a\}$
    - $Q^M = \{a\}$

$$\left.\begin{array}{l} \blacksquare \quad M \vDash \forall x\big(P(x) \rightarrow Q(x)\big),\ \exists x\ P(x) \\ \blacksquare \quad M \nvDash\ \forall x Q(x) \end{array}\right\} M \text{ is a counterexample}$$

# Thank You



https://xkcd.com/1033/